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EFFECT OF TRANSVERSE SHEAR AND ROTATORY INERTIA ON THE FORCED MOTION OF A STEPPED RECTANGULAR BEAM

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Forced motion of a rectangular beam whose thickness, density and elastic properties along the length vary in any number of steps is analyzed by the eigenfunction method using shear theory. A beam of two steps, clamped at both the edges and subjected to constant or half-sine pulse load is considered as an example problem. Numerical results computed for transverse defection are compared with those of classical theory.

1. INTRODUCTION

The free vibration of stepped beams has been analyzed by many researchers [1–13]. Filipich *et al*. [14] have considered the transverse vibration of a stepped beam subjected to an axial force and embedded in a non-homogeneous Winkler foundation. Bepat and Bhutani [15] have analyzed the free and forced vibration of stepped systems governed by a one dimension wave equation with non-classical boundary conditions. The authors are not aware of any paper on forced motion of beams of stepped thickness except that of the present authors [16].

In the present paper, the effect of transverse shear and rotatory inertia on the forced motion of a rectangular beam whose thickness, density and elastic properties along the length vary in any number of steps, is analyzed. The beam is assumed to be made up of n beam elements joined edge to edge and having in general different constant thickness, densities and elastic properties. Their free vibrations are considered using shear theory. The forced motion is analyzed by the eigenfunction method [17]. A beam made up of three beam elements, clamped at both edges and subjected to constant or half pulse load is considered as an example problem. The variations in lengths, thicknesses and densities of the elements are taken in such a way that the total length, average thickness and average density of the beam remain constant. Numerical results computed for the transverse deflection for various parameters of the beam are compared with those of classical theory.

2. EQUATION OF MOTION

An isotropic beam of breadth b and length a whose thickness, density and elastic properties along the length vary in steps is considered. The beam is defined by Cartesian co-ordinates by setting the x-axis along the length, the y-axis along the breadth, the middle plane of the beam in the plane $z = 0$ and the two edges in the planes $x = 0$ and $x = a$. The beam is assumed to be made up of n beam elements joined edge to edge with their middle planes lying in plane $z = 0$. The breadth, length, thickness, density, Young's

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modulus and Poisson's ratio of the kth element ($k = 1, 2, \ldots, n$) are taken as b, a_k, h_k, ρ_k, E_k and ν_k respectively and it lies from $x = x_{k-1}$ to $x = x_k$ where $x_k - x_{k-1} = a_k$, $x_0 = 0$ and $x_n = a$. Some of the thickness profiles of the beam along the length are shown in Figure 1.

The equations of motion of the beam elements according to shear theory are taken as

$$[E_k h_k^3/12(1-\nu_k^2)]\psi_{k,xx} - [K_s E_k h_k/2(1+\nu_k)](\psi_k + w_{k,x}) - \rho_k h_k^3 \psi_{k,tt} = 0,$$

$$K_s E_k h_k/2(1+\nu_k)(\psi_{k,x} + w_{k,xx}) - \rho_k h_k w_{k,tt} + p_k(x,t) = 0,$$

$$x_{k-1} \leqslant x \leqslant x_k, \quad k = 1, 2, \ldots, n \tag{1}$$

where w_k, ψ_k, p_k, t and K_s are the transverse deflections, rotations of the normal to the middle plane of the beam, loads per unit length, time and shear constant, respectively. A comma followed by a variable suffix denotes differentiation with respect to that variable.

Making the equations (1) non-dimensional,

$$I_k \psi_{k,XX} - L_k(\psi_k + W_{k,X}) - (\gamma_k H_k^3/12)\psi_{k,TT} = 0,$$

$$L_k(\psi_{k,X} + W_{k,XX}) + \gamma_k H_k W_{k,TT} + P_k(X,T) = 0, \qquad X_{k-1} \leqslant X \leqslant X_k, \tag{2}$$

where

$$X = x/a, \quad X_k = x_k/a, \quad H_k = h_k/a, \quad \gamma_k = \rho_k/\rho_a, \quad \varepsilon_k = E_k/E, \quad P_k = p_k/E,$$

$$T = t\sqrt{E/(\rho_a a^2)}, \quad I_k = \varepsilon_k H_k^3/12(1-\nu_k^2), \quad L_k = K_s \varepsilon_k H_k/2(1+\nu_k), \quad X_0 = 0,$$

$$X_n = 1.$$

ρ_a is the average density of the beam and E is the Young's modulus of some standard material.

3. FREE VIBRATION ANALYSIS

3.1. SOLUTION

For free vibration one takes

$$W_k(X,T) = W_{kj}(X)\,e^{i\Omega_j T}, \qquad \psi_k(X,T) = \psi_{kj}(X)\,e^{i\Omega_j T} \tag{3}$$

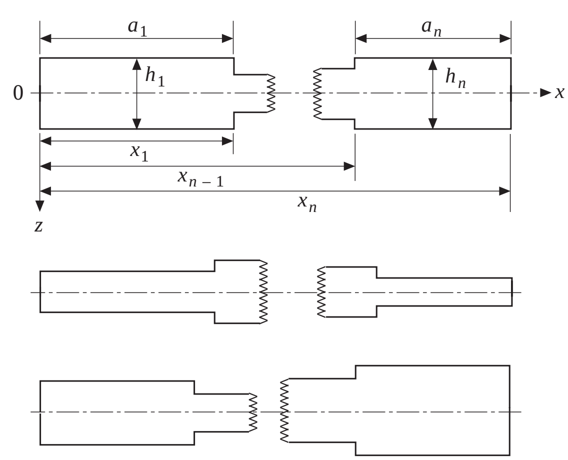

Figure 1. Thickness profiles of the beam.

and by substituting in equations (2) after putting $P_k = 0$ one gets,

$$I_k \psi_{kj,XX} - L_k(\psi_{kj} + W_{kj,X}) - (\gamma_k H_k^3/12)\Omega_j^2 \psi_{kj} = 0,$$
$$L_k(\psi_{jk,X} + W_{kj,XX}) + \gamma_k H_k W_{kj} \Omega_j^2 = 0, \tag{4}$$

where W_{kj}, ψ_{kj} are the mode shape functions and Ω_j is the circular frequency for the jth normal mode of free vibration.

For the sake of convenience, by suppressing the subscript j in the free vibration analysis and putting $(W_k, \psi_k) = (\bar{W}_k, \bar{\psi}_k)\,e^{\lambda_k X}$ in equations (4) and then eliminating $\bar{W}_k$ and $\bar{\psi}_k$ from them, one gets

$$12L_k I_k \lambda_k^4 + \gamma_k H_k(L_k H_k^2 + 12I_k)\Omega^2\lambda_k^2 + \gamma_k H_k(\gamma_k H_k^3 \Omega^2 - 12L_k)\Omega^2 = 0. \tag{5}$$

If λ_{1k}^2 and $-\lambda_{2k}^2$ are the roots of equation (5), then the solution of the equations (4) can be taken as

$$W_k(X) = G_k(X)D_k, \qquad \psi_k(X) = S_k(X)D_k, \tag{6}$$

where

$$D_k = [d_{1k} \quad d_{2k} \quad d_{3k} \quad d_{4k}]', \qquad G_k(X) = [\cosh\lambda_{1k}X \quad \sinh\lambda_{1k}X \quad \cos\lambda_{2k}X \quad \sin\lambda_{2k}X],$$
$$S_k(X) = [C_{1k}\sinh\lambda_{1k}X \quad C_{1k}\cosh\lambda_{1k}X \quad C_{2k}\sin\lambda_{2k}X \quad -C_{2k}\cos\lambda_{2k}X],$$
$$C_{1k} = -[L_k\lambda_{1k}^2 + \gamma_k H_k \Omega^2]/L_k\lambda_{1k}, \qquad C_{2k} = [L_k\lambda_{2k}^2 - \gamma_k H_k\Omega^2]/L_k\lambda_{2k}.$$

D_k is the vector of mode shape constants and the prime denotes the transpose of a matrix.

The continuity conditions between the line elements at $X = X_k$; $k = 1, 2, \ldots, n-1$ can be taken as

$$W_\ell(X_k) = W_k(X_k), \qquad \psi_\ell(X_k) = \psi_k(X_k), \qquad I_\ell \psi_{\ell,X}(X_k) = I_k \psi_{k,X}(X_k),$$
$$L_\ell\{W_{\ell,X}(X_k) + \psi_\ell(X_k)\} = L_k\{W_{k,X}(X_k) + \psi_k(X_k)\}, \tag{7}$$

where $\ell = k + 1$.

From equations (6) and (7) one gets

$$D_\ell = A^{(\ell)}D_k, \qquad A^{(\ell)} = A_\ell^{-1}(X_k)A_k(X_k), \tag{8}$$

where the matrices $A_k(X_k)$ and $A_\ell(X_k)$ are given by

$$A_k(X_k) = [G_k(X_k) \quad S_k(X_k) \quad I_k S_k(X_k) \quad L_k\{G_{k,X}(X_k) + S_k(X_k)\}]',$$
$$A_\ell(X_k) = [G_\ell(X_k) \quad S_\ell(X_k) \quad I_\ell S_\ell(X_k) \quad L_\ell\{G_{\ell,X}(X_k) + S_\ell(X_k)\}]'. \tag{9}$$

From equation (8), one gets

$$D_\ell = B^{(\ell)}D_1, \qquad B^{(\ell)} = A^{(\ell)}A^{(\ell-1)}\cdots A^{(2)} = [b_{qr}^{(\ell)}]_{4\times 4}. \tag{10}$$

In this way the $4n$ constants arising in solutions (6) reduce to 4. It can be seen that if the thicknesses, densities and elastic properties of the n beam elements are taken as the same, the matrices $A^{(\ell)}$ and $B^{(\ell)}$ reduce to unit matrices and the whole problem reduces to that of a uniform beam.

3.2. EDGE CONDITIONS

The beam is taken to be clamped at both edges, for which the conditions are

$$W_1(0) = \psi_1(0) = W_n(1) = \psi_n(1) = 0. \tag{11}$$

3.3. FREQUENCY EQUATION

Using relations (10) in solutions (6) and then putting them in conditions (11), we get

$$d_{11}+d_{31}=0, \qquad C_{11}\,d_{21}-C_{21}\,d_{41}=0,$$
$$s_{11}\,d_{11}+s_{12}\,d_{21}+s_{13}\,d_{31}+s_{14}\,d_{41}=0, \qquad s_{21}\,d_{11}+s_{22}\,d_{21}+s_{23}\,d_{31}+s_{24}\,d_{41}=0, \tag{12}$$

where

$$s_{1r}=G_n\,(1)\,[b_{qr}^{(n)}]_{4\times 1}, \qquad s_{2r}=S_n\,(1)\,[b_{qr}^{(n)}]_{4\times 1}, \qquad r=1,2,3,4. \tag{13}$$

For a non-trivial solution of equations (12) the determinant of the coefficient matrix must vanish, which gives rise to the following transcendental frequency equation

$$(C_{11}\,s_{24}+C_{21}\,s_{22})\,(s_{13}-s_{11})+(C_{11}\,s_{14}+C_{21}\,s_{21})\,(s_{21}-s_{23})=0. \tag{14}$$

The denumerable infinity of roots of this equation for given dimensions, densities and elastic constants of the beam elements are frequencies Ω_j of various normal modes of free vibration of the beam.

3.4. ORTHONORMALITY CONDITION

The orthogonality condition for normal modes of free vibration of the beam can be obtained as

$$\sum \gamma_k\,H_k \int_{x_{k-1}}^{x_k} [12W_{ki}\,W_{kj}+H_k^2\,\psi_{ki}\,\psi_{kj}]\,\mathrm{d}X=12\delta_{ij}, \tag{15}$$

where δ_{ij} is the Kronocker delta and summation over k is taken from 1 to n.

3.5. MODE SHAPES

Since out of the four equations (12) only three are independent, one solves first three of them to get D_1 in terms of d_{41}. This is substituted in equations (10) to get D_2 and D_3 in terms of d_{41}. These are then substituted in solutions (6) to get the mode shapes as

$$W_k\,(X)=G_k\,(X)\,[e_{1k}\quad e_{2k}\quad e_{3k}\quad e_{4k}]'d_{41}, \qquad \psi_k\,(X)=S_k\,(X)\,[e_{1k}\quad e_{2k}\quad e_{3k}\quad e_{4k}]'d_{41},$$
$$X_{k-1}\leqslant X\leqslant X_k, \quad k=1,2,\ldots,n. \tag{16}$$

where

$$d=(C_{21}\,s_{12}+C_{11}\,s_{14})/C_{11}\,(s_{11}-s_{13}), \qquad e_{11}=-d, \qquad e_{21}=C_{21}/C_{11}, \qquad e_{31}=d,$$
$$e_{41}=1, \qquad e_{q\ell}=d(b_{q3}^{(\ell)}-b_{q1}^{(\ell)})+(C_{11}\,b_{q4}^{(\ell)}+C_{21}\,b_{q2}^{(\ell)})/C_{11}, \qquad q=1,2,3,4. \tag{17}$$

To get d_{41}, normalization condition (15) is used to give

$$d_{41}^2=1/\sum [F_k\,(X_k)-F_k\,(X_{k-1})], \tag{18}$$

where

$$F_k\,(X)=\gamma_k\,H_k\,[f_{1k}\,X+f_{2k}\,\sinh(2\lambda_{1k}\,X)+f_{3k}\,\sin(2\lambda_{2k}\,X)$$
$$+f_{4k}\,\cosh(2\lambda_{1k}\,X)+f_{5k}\,\cos(2\lambda_{2k}\,X)+\vartheta_k\,[\cosh(\lambda_{1k}\,X)\{f_{6k}\,\sin(\lambda_{2k}\,X)$$
$$+f_{7k}\,\cos(\lambda_{2k}\,X)+\sinh(\lambda_{1k}\,X)\{f_{8k}\,\sin(\lambda_{2k}\,X)+f_{9k}\,\cos(\lambda_{2k}\,X)\}]], \tag{19}$$
$$\vartheta_k=2/(\lambda_{1k}^2+\lambda_{2k}^2), \qquad 2\lambda_{1k}\,f_{1k}=h_{1k}\,(e_{1k}^2-e_{2k}^2)+h_{2k}\,(e_{3k}^2+e_{4k}^2),$$

$$4\lambda_{1k} f_{2k} = h_{3k}(e_{1k}^2 + e_{2k}^2), \qquad 4\lambda_{2k} f_{3k} = h_{4k}(e_{3k}^2 - e_{4k}^2), \qquad 2\lambda_{1k} f_{4k} = h_{3k} e_{1k} e_{2k},$$

$$2\lambda_{2k} f_{5k} = -h_{4k} e_{3k} e_{4k}, \qquad f_{6k} = 12 g_{1k} + h_{5k} g_{2k}, \qquad f_{7k} = 12 g_{3k} - h_{5k} g_{4k},$$

$$f_{8k} = 12 g_{4k} + h_{5k} g_{3k}, \qquad f_{9k} = 12 g_{2k} - h_{5k} g_{1k}, \qquad g_{1k} = \lambda_{1k} e_{2k} e_{4k} + \lambda_{2k} e_{1k} e_{3k},$$

$$g_{2k} = \lambda_{1k} e_{1k} e_{3k} - \lambda_{2k} e_{2k} e_{4k}, \qquad g_{3k} = \lambda_{1k} e_{2k} e_{3k} - \lambda_{2k} e_{1k} e_{4k},$$

$$g_{4k} = \lambda_{1k} e_{1k} e_{4k} + \lambda_{2k} e_{2k} e_{3k}, \qquad h_{1k} = (12 - H_k^2 C_{1k}^2), \qquad h_{2k} = (12 + H_k^2 C_{2k}^2),$$

$$h_{3k} = (12 + H_k^2 C_{1k}^2), \qquad h_{4k} = (12 - H_k^2 C_{2k}^2), \qquad h_{5k} = C_{1k} C_{2k} H_k^2. \tag{20}$$

4. FORCED MOTION ANALYSIS

A solution of the forced motion equations (2) subjected to the continuity conditions (7) and edge conditions (11) is assumed to be

$$W_k(X, T) = \sum W_{kj}(X) g_j(T), \qquad \psi_k(X, T) = \sum \psi_{kj}(X) g_j(T),$$

$$X_{k-1} \leqslant X \leqslant X_k, \quad k = 1, 2, \ldots, n, \tag{21}$$

where the summation over j is from 1 to ∞. By substituting it in equations (2) and using equations (4), one gets

$$\sum \gamma_k H_k W_{kj}(g_{j,TT} + \Omega_j^2 g_j) = P_k(X, T), \qquad \sum (\gamma_k H_k^3/12) \psi_{kj}(g_{j,TT} + \Omega_j^2 g_j) = 0. \tag{22}$$

Using equations (22) and the orthonormality condition, one gets

$$g_{j,TT} + \Omega_j^2 g_j = G_j(T), \tag{23}$$

where

$$G_j(T) = \sum \int_{x_{k-1}}^{x_k} P_k W_{kj} \, \mathrm{d}X. \tag{24}$$

The solution of equation (23) is

$$\Omega_j g_j(T) = \Omega_j g_j(0) \cos(\Omega_j T) + g_{j,T}(0) \sin(\Omega_j T) + \int_0^T G_j(\tau) \sin\{\Omega_j(T - \tau)\} \, \mathrm{d}\tau, \tag{25}$$

where

$$g_j(0) = \sum \gamma_k H_k \int_{x_{k-1}}^{x_k} W_k(X, 0) W_{kj} \, \mathrm{d}X, \qquad g_{j,T}(0) = \sum \gamma_k H_k \int_{x_{k-1}}^{x_k} W_{k,T}(X, 0) W_{kj} \, \mathrm{d}X. \tag{26}$$

If the initial conditions are taken as

$$W_k(X, 0) = W_{k,T}(X, 0) = 0,$$

then

$$g_j(0) = g_{j,T}(0) = 0. \tag{27}$$

4.1. LOADING CONDITION

The following two types of external loads uniformly distributed over a portion of each beam element are taken:

4.1.1. *Constant load* (*CL*)

$$P_k\,(X, T) = P_0\,[U(X - \xi_k) - U(X - \eta_k)]U(T)/\sum(\eta_k - \xi_k);$$

$$X_{k-1} \leqslant \xi_k < \eta_k \leqslant X_k, \quad k = 1, 2, \ldots, n. \tag{28}$$

where P_0 is the total load on the beam.

$G_j\,(T)$, evaluated after substituting from equations (16) and (28) in equation (24), is substituted in equation (25) and the condition (27) is used, to get

$$g_j\,(T) = P_j\,[1 - \cos(\Omega_j\,T)]/\Omega_j^2, \tag{29}$$

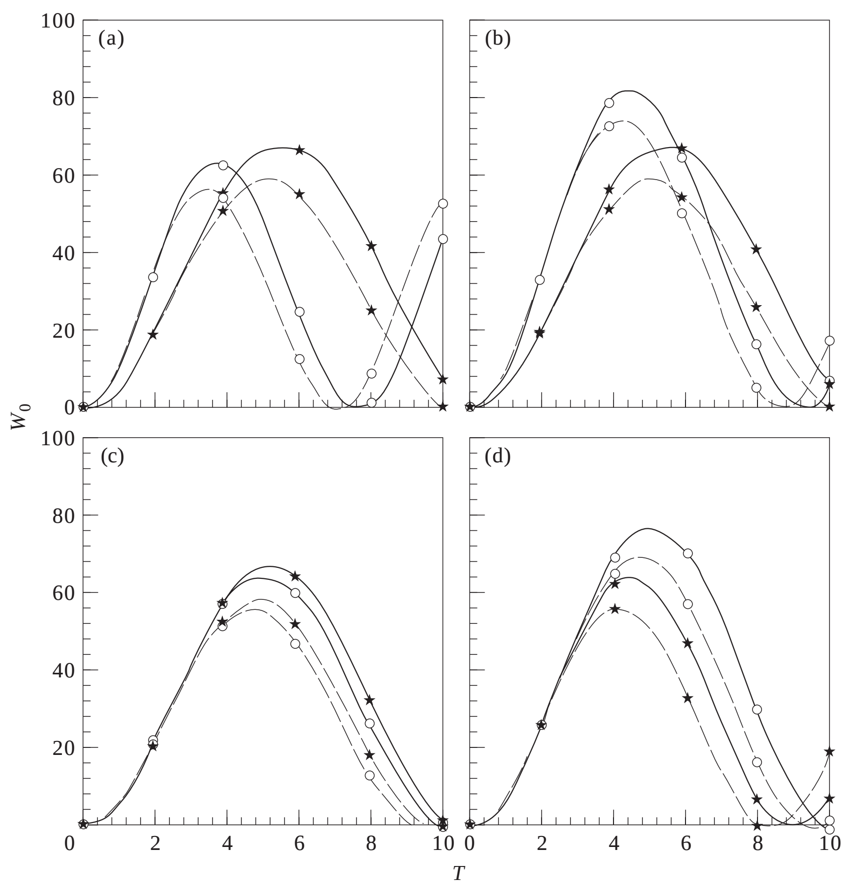

Figure 2. W_0 versus T for CL for various values of β_2 and α_2; ——, shear theory; ------, classical theory; (a) $\alpha_2 = 1{\cdot}3$, $\delta_2 = \varepsilon_2 = 1{\cdot}0$; ○, $\beta_2 = 0{\cdot}4$; ★, $\beta_2 = 1{\cdot}6$. (b) $\alpha_2 = 0{\cdot}7$; $\delta_2 = \varepsilon_2 = 1{\cdot}0$; ○, $\beta_2 = 0{\cdot}4$; ★, $\beta_2 = 1{\cdot}6$. (c) $\beta_2 = 1{\cdot}3$, $\delta_2 = \varepsilon_2 = 1{\cdot}0$; ○, $\alpha_2 = 0{\cdot}4$; ★, $\alpha_2 = 1{\cdot}6$. (d) $\beta_2 = 0{\cdot}7$, $\delta_2 = \varepsilon_2 = 1{\cdot}0$; ○, $\alpha_2 = 0{\cdot}4$; $\alpha_2 = 1{\cdot}6$.

where

$$P_j = P_0 \sum [\phi_{kj}(\eta_k) - \phi_{kj}(\xi_k)] / \sum (\eta_k - \xi_k),$$

$$\phi_{kj}(X) = d_{41j}\,[\{e_{1kj}\sinh(\lambda_{1kj}X) + e_{2kj}\cosh(\lambda_{1kj}X)\}/\lambda_{1kj} + \{e_{3kj}\sin(\lambda_{2kj}X) - e_{4kj}\cos(\lambda_{2kj}X)\}/\lambda_{2kj}]. \quad (30)$$

4.1.2. *Half sine pulse load* (*HL*)

$$P_k(X, T) = P_0\,[U(X-\xi_k) - U(X-\eta_k)]\{1 - U(T-t_1)\}\sin(\pi T/t_1)/\sum(\eta_k - \xi_k);$$

$$X_{k-1} \leqslant \xi_k < \eta_k \leqslant X_k, \quad k = 1, 2, \ldots, n. \quad (31)$$

where t_1 is the duration of HL.

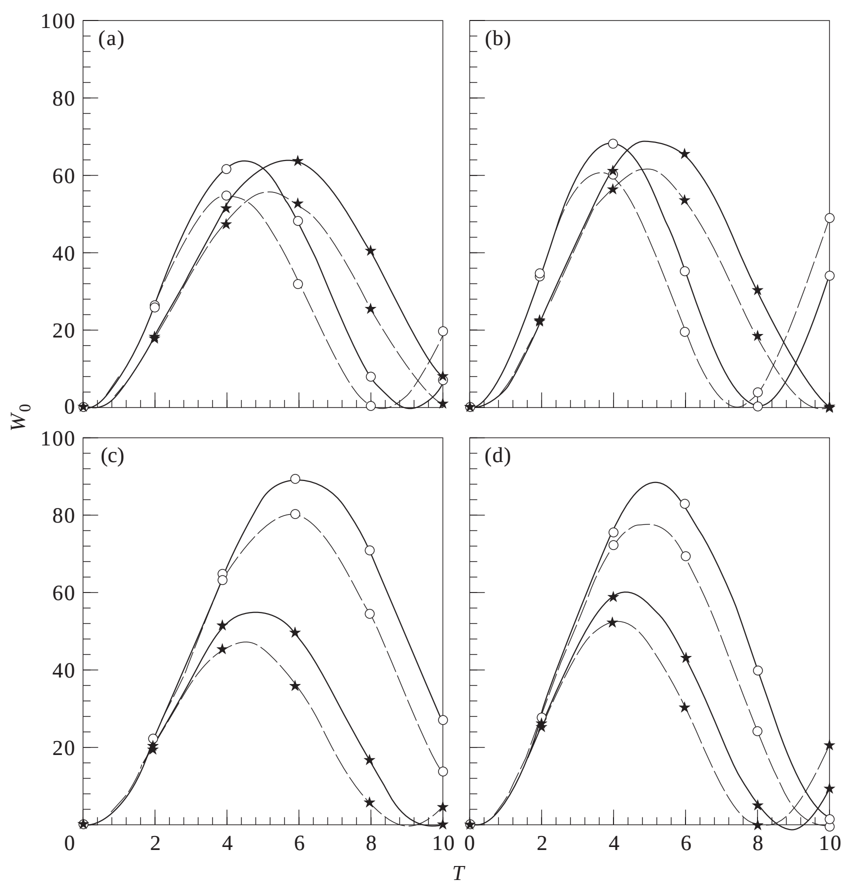

Figure 3. W_0 versus T for CL for various values of δ_2 and ε_2. Key as Figure 2 except (a), (b) ○, $\delta_2 = 0{\cdot}4$; ★, $\delta_2 = 1{\cdot}6$. (c), (d) ○, $\varepsilon_2 = 0{\cdot}4$; ★, $\varepsilon_2 = 1{\cdot}6$.

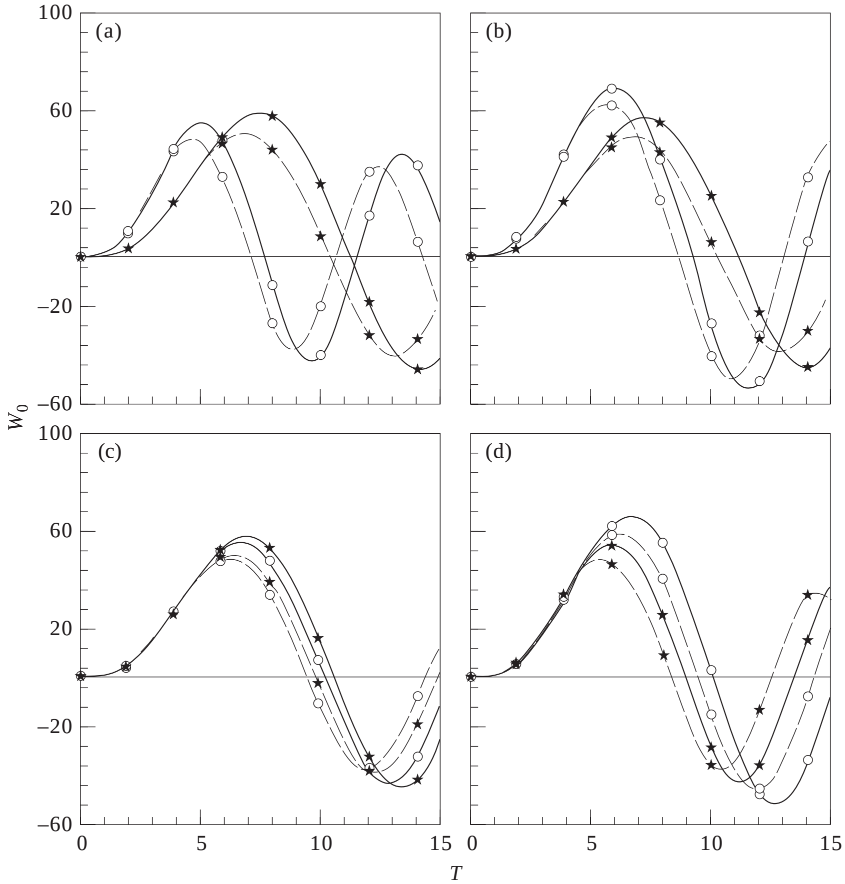

Figure 4. W_0 versus T for HL for various values of β_2 and α_2. Key as Figure 2.

By proceeding as above, one gets

$$g_j(T) = \begin{cases} P_j\, t_1\, [\pi \sin(\Omega_j T) - \Omega_j t_1 \sin(\pi T/t_1)]/[\Omega_j(\pi^2 - \Omega_j^2 t_1^2)], & \text{when} \quad T < t_1 \\ 2P_j\, \pi t_1\, [\sin\{\Omega_j(T - t_1/2)\} \cos(\Omega_j t_1/2)]/[\Omega_j(\pi^2 - \Omega_j^2 t_1^2)], & \text{when} \quad T \geqslant t_1 \end{cases}. \tag{32}$$

The substitution of unique mode shapes W_{kj} given by equations (16) and (18) and $g_j(T)$ from equation (29) or (32) as the case may be, gives the transverse deflection $W_k(X, T)$ for forced motion.

5. RESULTS AND DISCUSSION

The variations in lengths, thicknesses and densities of different beam elements are taken in such a way that the total length, average thickness and average density of the beam remain constant, thus

$$\alpha_k = a_k/a_1, \qquad \beta_k = h_k/h_1, \qquad \delta_k = \rho_k/\rho_1. \tag{34}$$

Now $\Sigma a_k = a$ or $a_1 \Sigma\alpha_k = a$ or $X_1 = 1/\Sigma\alpha_k$ and $X_k = X_1 \Sigma_{i=1}^{k} \alpha_i$; $\Sigma a_k h_k = a h_a$ or $a_1 h_1 \Sigma\alpha_k \beta_k = a h_a$ or $H_1 = H_a/(X_1 \Sigma\alpha_k \beta_k)$ and $H_k = H_1 \beta_k$, where h_a is the average thickness of the beam and $H_a = h_a/a$; $\Sigma a_k h_k \rho_k = a h_a \rho_a$ or $\gamma_1 = H_a/(X_1 H_1 \Sigma\alpha_k H_k \delta_k)$ and $\gamma_k = \gamma_1 \delta_k$.

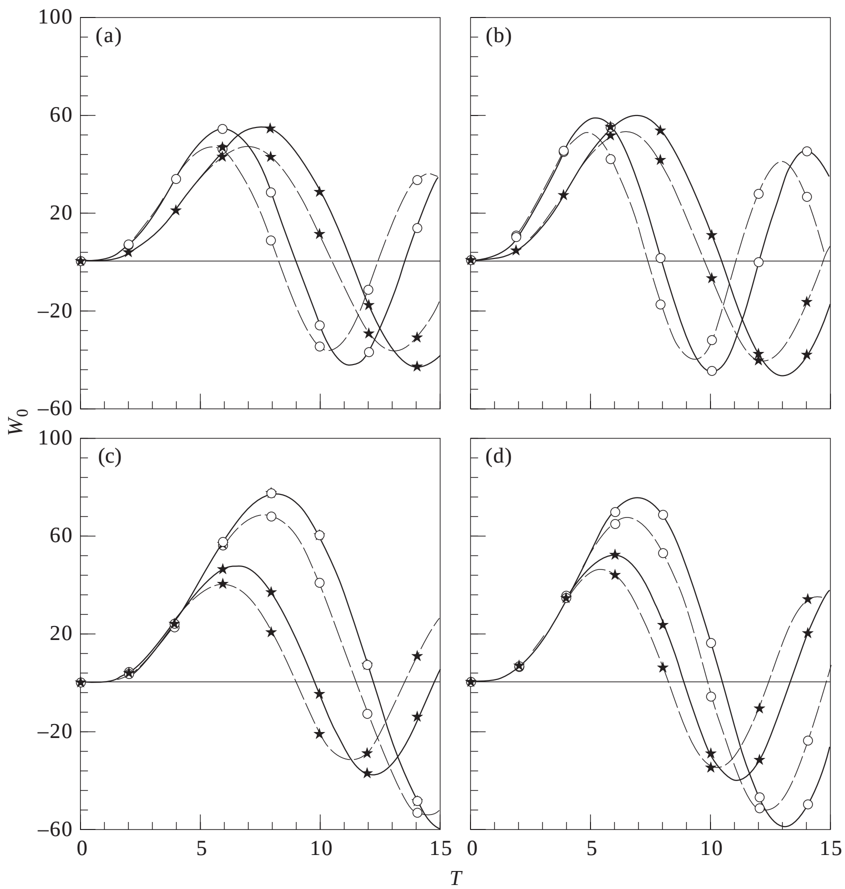

Figure 5. W_0 versus T for HL for various values of δ_2 and ε_2. Key as Figure 3.

Numerical results are computed for the transverse deflection parameter $W_0 = (W_k/P_0)_{x=0\cdot5}$ for a beam made up of three beam elements whose first and third elements are identical i.e., for $\alpha_3 = \beta_3 = \delta_3 = \varepsilon_3 = 1$, by taking $\nu_1 = \nu_2 = \nu_3 = 0\cdot3$ and $H_a = 0\cdot1$.

The frequencies Ω_j are computed by the bisection method up to an accuracy of five decimal places and the series of W_k (equation (21)) is summed to the first ten terms which gives an accuracy of four decimal places.

The graphs of W_0 versus T for CL and HL for various values of β_2, α_2, δ_2 and ε_2 are plotted in Figures 2–5 for shear theory as well as for classical theory. It can be seen in all cases, that the magnitude of W_0 at the first peak and the time of attaining the first peak is higher in shear theory.

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